Beyond Formulas in Mathematics and Teaching

DYNAMICS OF THE HIGH SCHOOL ALGEBRA CLASSROOM

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Foreword by Penelope Peterson

TEACHERS COLLEGE PRESS

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New York and London

students trusted their own judgment and experience and were willing to question my views. In mathematics, when we were focused on the content, we did not have real conversations very often; at most there was a tutorial form of dialogue. In other classes, there were conversations: students talked to the group, not just to me. Ironically, given traditional arguments for the ways in which mathematics improves people's reasoning skills and critical faculties, when contrasted with other classes, my mathematics classes did not seem like a context that would foster such faculties or skills.

I wondered whether the difference in discourse patterns was a simple reflection of the nature of the subject matter. In traditional school instruction, there is a strong dichotomy between matters of opinion or interpretation and knowledge. According to this view, given the nature of the material, it is natural to see differences between classes. Algebra is not about interpretation; algebra is about facts. In mathematics classes, students' statements are either "right" or "wrong." In my other classes, students' interpretation of a text was rarely "wrong"; instead, if it was "off the mark," it was "unlikely" or "unconvincing."

I also wondered whether the differences I was noticing were somehow intrinsically linked to different subject matters and their role in schooling. Acceptance to college would be influenced by students' scores on a multiple-choice mathematics test. Maybe this explained the difference. But neither of these potential explanations made teaching mathematics feel any better.

An Illustration of My Teaching at This Time

To illustrate how I taught mathematics at this time, I'll examine, in some detail, how I taught one lesson in Chapter 10 on page 380 of the Dolciani and Wooton (1970/1973) text, a lesson titled "Determining an Equation of a Line." Though this example may bring back painful memories for some readers, I think it is important to illustrate the kind of teaching I used to do (which is still prevalent in many algebra classrooms). Though I will do my best to explicate the method that I taught, the details of the method are not crucial. It is more important to characterize the kind of teaching that I did at this time and the nature of the tasks students faced.

I am choosing the skill of writing an equation of a line through two points as an illustration of my teaching at this time because of its continued importance in school mathematics; it appears even in algebra textbooks that downplay the manipulation of symbols. (For example, this method appears in the University of Chicago School Mathematics Program's algebra text, McConnell et al., 1990.) Perhaps it continues to appear because, in statistics and in analytic geometry, students are expected to be able to write the equation of a line through two points. But in my early teaching, like many other mathematics teachers, I did not help students learn why this skill is considered important in school mathematics. I did not teach students how this line represents a constant rate of change, let alone help them examine purposes for writing the equations of lines. I just taught the method.

To be more specific, we did not explore how, if one assumes a constant rate of growth, a line can be used to interpolate or extrapolate from two data points—representing the coordinated values of two quantities—and make predictions. We did not discuss, for example, how this kind of extrapolation is routinely used in surveys for predicting some characteristic of a population based on responses of a small sample.

This kind of assumption of linearity to make predictions is made in other situations as well. Perhaps an example will clarify. If we are given two measurements of the length of an infant, one on its 7th day and one on its 21st day, it seems natural to try to compute the rate at which the baby is growing—the average number of inches it is growing per day—by taking the total amount of growth and dividing it by the number of days elapsed. We can use the resulting number to interpolate—to determine likely lengths of the infant on intervening days—and to extrapolate—to predict the likely length of the baby in the future.

In computing this growth rate and using it to make predictions, we have made some important assumptions. Recent research, and much folk wisdom, suggests that infants grow in spurts (a large amount of growth one day and almost no growth another), that this process is not even constant across a week's time. Yet we are deciding to assume that babies' growth in length can be approximated with a constant rate. In doing so, it is as if we are saying "Don't bother me with the day-to-day details." We imagine a situation in which babies grow at a constant rate. We postulate the equal sharing of change in length across equally sized units of time, even though we know this is not true. We hope that the predictions that we make using this strategy will be "close enough." I did not talk about this sort of prediction in my class or debate with students when it might be used profitably, and when it might be inappropriate.

Also, I am struck, in retrospect, by the complexity of the method I taught. The reasons that this complex method works seem quite complicated. It is also striking to me that I taught only one method and had no alternatives.

Background to the Method. Before presenting the complicated method I used to teach, for those who have not been thinking about algebra recently, I'll begin with some background. In traditional algebra instruction, students do most of their work with x's and y's. But there is another mathematical representation that also appears, graphs (straight lines and curves) in the Cartesian, or x-y, plane. Points in the Cartesian plane are located and named with a pair of numbers or coordinates that represent their "signed" horizontal and vertical distance from a point of origin. For the points (5, 2) and (2, −1) in the Cartesian plane, the first numbers represent right/left "signed" distance from the point of origin and the second numbers the up/down distance from that point.

The Cartesian plane and the x's and y's of algebra are linked in the following way: x's stand for the first coordinate (the right/left distance from the origin) and the y's stand for the second coordinate. Thus, an equation in x's and y's can represent a graph. In particular, when a straight line graph is described in the Cartesian plane, students are taught to write its "equation" (commonly, if superficially, defined by its form as a symbol string with one equal sign). In teaching this lesson from Dolciani and Wooton's (1970/1973) text, I taught students how to find the equation in "slope-intercept" form for the line through two points in a Cartesian plane (e.g., Find the equation of the line through the points (5, 2) and (2, −1)).

For the line through the points (5, 2) and (2, −1), the expected equation would be $y = x - 3$. This equation is related to the line in the following sense: All the paired values of x and y that make this equation a true statement are the coordinates of points along this line; if you subtract three from the x-coordinate, you get the y-coordinate. Most important in terms of the goal of the problem, 2 is equal to $5 - 3$ and -1 is equal to $2 - 3$, so both (5, 2) and (2, −1) lie on the line (see Figure 1.1).

The Method. Given the coordinates of two points—I will continue to use (5, 2) and (2, −1) as an example—students were to go through four steps, one of which is downplayed in the text. This four-step procedure (see Figure 1.2) involves other procedures taught in different chapters in the book.

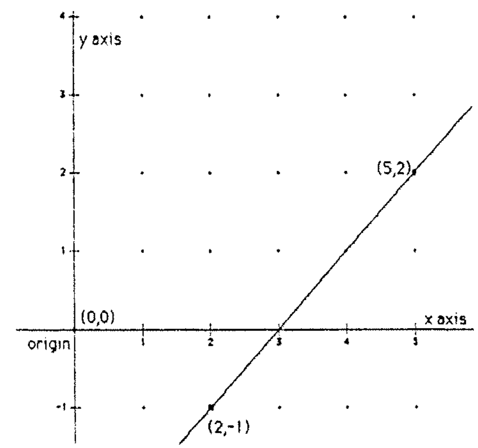
First, in the downplayed step, students were expected to realize that they were being asked to write an equation for a line in the "slope-intercept form," $y = mx + b$. For readers not fluent with this terminology, in this form, m stands for the slope of the line. The word *slope* describes the graph visually in terms of rise and run; a steeper line has more "rise" per unit of "run." But this number also represents the rate of change of y in terms of change in x. Typically, slope is reported as a ratio, rise/run; creating this ratio and carrying out the division allows one to compare different lines by finding the amount of rise for one unit of run. b stands for the value of y-coordinate of the point where the line crosses the y-axis, the "y-intercept" of the line (see Figure 1.3).[3] x and y for their respective coordinates for the points on the line described by the equation. Thus, this form is the "slope intercept form," or more appropriately "the slope y-intercept form." (See Chazan, 1995, for consideration of two other forms for writing linear functions.)

Students were to choose this form, even though the two given points do not automatically provide information on the slope and the y-intercept, because the slope-intercept form had been introduced in the previous section. Earlier in the chapter a different form of an equation for a line had been given, but that form (which in this case would give the equation $-x + y = -3$) was not to be used here.

Second, students were to compute the slope from the information given about the two points, a task familiar from earlier in the chapter. They were to do this by subtracting the y-coordinates of the two points to get the total rise ($2 - -1 = 3$), subtracting the x-coordinates ($5 - 2 = 3$) to get the total run, and then creating a ratio of rise/run ($3/3 = 1$). So m, the slope, is equal to one; along this line for every one unit of run the graph rises one unit. Since, in my teaching at the time, all of this work to compute the slope was being done without a graph, students did not use visual strategies, like counting over, for determining rise, run, and slope.

The third step, finding the y-intercept, which regularly confused my day-school students, was not justified in the text and is difficult to explain succinctly. At this point in the method, the slope has been determined, but the y-intercept has not been. In the Cartesian plane, the book's strategy is the equivalent of fixing a line through one of the two points with the correct slope and then

Figure 1.1. Representing $y = x - 3$.



The graph of $y = x - 3$

x-coordinates	y-coordinates
0	-3
1	-2
2	-1
3	0
4	1
5	2
6	3
7	4
8	5
9	6
10	7

Coordinates of eleven points on the graph of $y = x - 3$

Figure 1.2. Finding the equation of a line.

EXAMPLE. **Find an equation of the line which passes through the points whose coordinates are (5, 2) and (2, −1).**

Solution:

1. Slope $= m = \dfrac{-1-2}{2-5} = \dfrac{-3}{-3} = 1$

2. The slope-intercept form of the equation is $y = mx + b$**. Thus:**

$$y = 1x + b$$

Choose one point, say (5, 2). Since it lies on the line:

$$2 = 1 \cdot 5 + b, \quad \text{or} \quad 2 = 5 + b$$

$$\therefore -3 = b.$$

3. To check, show that the coordinates of the other point (2, −1) satisfy the equation:

$$y = x - 3$$
$$-1 \stackrel{?}{=} 2 - 3$$
$$-1 = -1 \checkmark$$

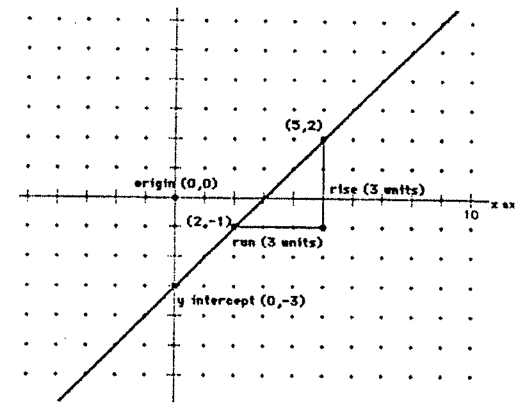
$\therefore$ **an equation of the line is** $y = x - 3$**. Answer.**

Reprinted with permission from Dolciani & Wooton (1970/73)

determining where it crosses the y-axis (where the x-coordinate is 0). In order to determine the y-intercept, students were to take the form $y = mx + b$, substitute the value of the slope and the coordinates of one of the points, and create an equation in b (e.g., $2 = 1 * 5 + b$). They were then supposed to solve this equation for b (in this case, by subtracting five from both sides).[4] Having in this way determined the y-intercept, they were supposed to take the final value for b (−3) and substitute it into the y-intercept slot in the equation, while reintroducing x and y into their slots, giving the equation $y = 1 * x - 3$.

Finally, students were to check that this equation worked by substituting the coordinates of the second point and verifying that the equation worked. In this case, −1 is indeed equal to $1 * 2 - 3$.

Figure 1.3. Slope and y-intercept.



Reflection on This Illustration. What does the long exposition of my teaching of this method tell us about the nature of my dayschool students' experience of algebra? I suggest that teaching this procedure in this way fits with what Richard Skemp (1976) calls an instrumental view of mathematics. In this view, mathematics is a set of procedures (some of which are complicated) that must be mastered (not understood) for solving problems whose origin and purpose are not available to students.

For some of my students there seemed to be a feeling of accomplishment associated with mastering such procedures, but for many, learning this procedure was frustrating. The procedure is long; it is hard to keep the goal in mind as you tackle one of the subtasks. The students had mastered the skills required to solve the subtasks one at a time when we "covered" them, but usually had forgotten each skill just a few short weeks later. Their skills did not accumulate and become linked into the larger routines needed to solve more advanced problems like finding the equation of a line through two points. Furthermore, at each step, there are reasons to wonder why this method works or how anyone ever invented or discovered it.

In class, I would spend a few days helping students gain limited (and I might add short-lived) mastery of this procedure. But I did not engage students in an examination of purposes one might have for determining an equation for a line between two points.

We also did not talk about the difference between solving for a coefficient, or parameter (as in the third step), and the solving of equations "for x" that they had done before. In my teaching at that time, this complex procedure was taught divorced from the reasons that it works and consideration of anything but the textbook exercise of generating an equation for points in the Cartesian plane that were not intended as representations of other quantities.[5]

An Opportunity for Change

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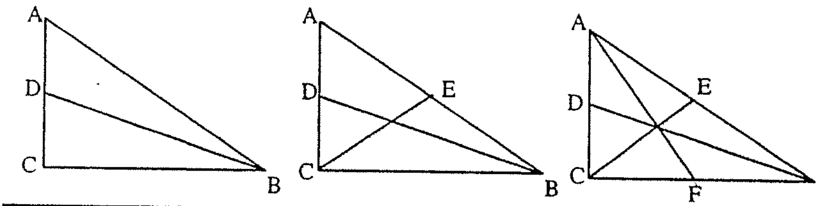
In my third and final year of teaching at this school, though I still did not know how to change the nature of the rationale for student engagement, I began to see how conversations in the mathematics classroom might be different, how I might be able to create a more student-centered classroom. I heard about a project that was looking for teachers willing to pilot the use of a new piece of software then under development for teaching geometry classes. I had a small class and access to a small number of Apple II computers; I decided to try and was encouraged by what I saw. While algebra class was still the same, we had more interesting conversations in geometry class. This difference was illuminating.

The Geometric Supposer (Schwartz et al., 1985; Schwartz & Yerushalmy, 1990) was developed to facilitate students' development of conjectures (or what George Polya, noted mathematician and mathematics educator, calls "educated guesses") by providing them with opportunities to explore geometrical constructions empirically, with measurement of particular cases. The notion underlying the design of the software is that access to the drawing and measurements provided by the computer helps students evaluate their own ideas. Rather than waiting for a text or a teacher to indicate that an idea is right or wrong, students can use exploration of particular cases to discover counterexamples that support or refute their ideas. If there are no counterexamples, there is good reason to think that the idea is true and to try to understand why by writing a deductive proof. Furthermore, students' opportunity to work with many diagrams on a regular basis can help develop their facility with diagrams. (For an examination of this claim, see Yerushalmy & Chazan, 1990.)

By examining diagrams and at measurements, my students were able to decide what relationships they thought were true for a particular construction. This had a big impact on our conversations in class. For example, I once asked students to draw one median, two medians, and then three medians in a triangle (a median is a segment from a vertex or corner to the midpoint of the opposite side, see Figure 1.4) and record conjectures they had about relationships between elements in the resulting diagrams. I did not hold out great hopes for the part of the problem about one median, but included it since they would have to pass through that stage in making the other medians.

Figure 1.4. Drawing medians in a triangle.

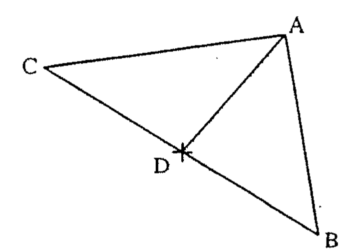


The students developed their own ideas, which we then discussed. Students had different ideas. One student, whom I'll call Rachel, had an idea that I had not seen before. Based on the triangle inequality that we had studied before (that the sum of the lengths of two sides of a triangle is always more than the length of the third side), she developed an argument, different from the proof in our book, that the length of any median to a side in any triangle is less than half the sum of the lengths of the other two sides (in Figure 1.5 the length of AD is less than half the sum of the lengths of AB and AC).

She doubled the median and extended it out past the side of the original triangle. I do not know what made her think to do that. When I asked, she could not really explain. She noticed that if she connected the endpoint of the extension to the vertices of the original triangle, she got a parallelogram (see Figure 1.6). If ABEC is a parallelogram, then its opposite sides (e.g., AC and BE or AB and CE) have equal lengths.

Then she looked at the diagram in a different way. She imagined that the parallelogram was split into two triangles by the segment AE; she focused on

Figure 1.5. One median in a triangle.



class, particularly the ways in which they insisted on sensible uses of mathematics in situated contexts (for an examination of such strengths, see Chazan, 1996a).

In addition, in lower-track secondary school mathematics classrooms, the notion of ability has an impact on the dynamics of classroom interaction. At first, the students that Sandy and I taught sometimes seemed reluctant to share their thinking with us. They had been placed in our class as a result of their achievement in prior courses; their achievement had been evaluated and found wanting. They were accustomed to the teacher as evaluator. We were teachers, members of a group that had previously judged them to be of "low ability." Perhaps they were concerned that if they shared their thinking, we might once again judge that they knew little.

Second, we often found that our students were not in the habit of listening to one another. This, again, seemed related to the notion of ability. It can be especially difficult to help students in a lower-track class see a purpose in listening to each other. If all the students in the room have been placed in this class because they have done poorly in mathematics, students may wonder why they should listen to each other's comments.

For teachers who want to create classroom conversation, these two dynamics result in a serious concern about student participation. In Chapter 2, I discussed Mary Metz's (1993) notion of the teacher's dependence on students. Yet her argument about the teacher's dependence on students' learning is situated in the context of traditional instruction. Student-centered teaching makes the teacher all the more dependent on students. Not only were we dependent on our students to learn, but we also depended on them to help produce classroom interaction, or our lessons could come to a grinding halt.

CHANGING THE ROLE OF THE "PROBLEM"

How does one begin to change the mathematics classroom to create different roles for teachers and students? The shifts called for by the NCTM *Professional Standards for Teaching Mathematics* (1991) are not widespread in secondary school mathematics classrooms, but there has been some progress in developing and instantiating alternative stances toward mathematical knowledge in the practice of elementary school mathematics instruction.[15] For example, Magdalene Lampert, an educator known for changing the nature of the interaction in her own elementary mathematics classroom, builds explicitly on views of mathematical practice. In a piece titled "When the Problem Is Not the Question and the Solution Is Not the Answer: Mathematical Knowing and Teaching" (1990) she describes her own classroom stance. As a teacher, Lampert steps away from the role of sole arbiter of mathematical truth and insists that students help decide what is true. Rather than have students memorize procedures for finding solutions, she asks students to conjecture and invent as they solve problems.[16] Instead of textbooks, homework, and fill-in-the-blank worksheets, her instruction includes a bound student notebook in which students write with pens so they will not erase their reasoning (one small indication to students of the importance placed on their reasoning); a format for organizing work in this notebook; the introduction and use of the terms *conjecture, hypothesis, revise, agree, disagree,* and *prove*; norms for the respect of others' ideas; and whole-class discussions as a central element.

However, in order to carry out these changes, there must be something to discuss; the contents of the lesson cannot be "cut and dried." To paraphrase Lampert (1990), "doing mathematics" can no longer mean following the rules laid down by the teacher; "knowing mathematics" can no longer mean remembering and applying the correct rule when the teacher asks a question; and "mathematical truth" can no longer be determined when the answer is ratified by the teacher (p. 32).

While Lampert makes many changes in her instruction, I will focus on the way she changes the role of the problem in mathematics instruction. On the surface, she seems to assign traditional problems/exercises to students; seemingly, like most of the problems assigned in school, her problems "are well formulated, present no ambiguity, admit a few objective solutions, and can be solved by the application of a suitable combination of learned algorithms" (Borasi, 1992, p. 167). But closer examination reveals this not to be true. As the title of Lampert's article suggests, in her classroom, "the problem is not the question and the solution is not the answer"; hidden beneath the seemingly cut-and-dried problem is an instructional design and a set of classroom expectations that elicit a multiplicity of student strategies.

In her teaching, Lampert changes the role of the problem by making a clever inversion. Rather than have mathematics problems follow direct instruction of algorithms that solve these problems and use problems solely as an assessment tool to determine whether students have learned the "covered" material, she assigns a problem before an algorithm for its solution is taught and uses student exploration of such problems as a central way for students to learn the mathematics she seeks to teach.[17] In typical instruction, once an efficient algorithm is taught, there is a press to use this algorithm rather than idiosyncratic solution methods. When a seemingly unambiguous problem is used before an algorithm has been taught, it is no longer cut-and-dried; it often can be approached in different and innovative ways.

In Lampert's instruction, student exploration of a problem comes prior to classroom discussion during which students share the results of their exploration. Through the discussion of their results, the class as a whole (the teacher included) develops a shared set of mathematical understandings. Since the problems she uses may have a limited number of answers, while students may develop a range of solution procedures, they must come to grips with each other's ideas as they come to consensus about the correct answers. It is the resulting conversation around the problem, students' solution methods, and their developing mathematical ideas that are important. Direct instruction, if carried out at all, might follow up on a discussion of student results, indicating how these results relate to accepted mathematical thought, or underlining the consensus that has developed in the group.

Thus, in her instruction, Lampert portrays school mathematics like the discipline itself, as a living and growing field in which developments occur when people create solutions to problems. She encourages her students to see mathematics as a field in which one makes hypotheses and revises them. Since she asks students to tackle problems before having taught them algorithmic solutions to problems of these kind, right and wrong are less central categories in her teaching than in traditional teaching; students are not expected to be able to come to correct answers in their first attempts at a problem of a particular kind. Incorrect answers are as deserving of classroom attention as correct ones (without implying that they are correct); what matters is the nature of students' mathematical reasoning and its evolution.

A SAMPLE HIGH SCHOOL ALGEBRA LESSON

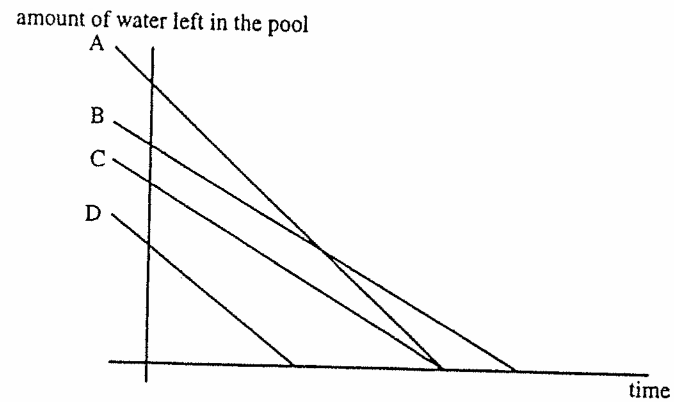
In "When the Problem Is Not the Question and the Solution Is Not the Answer: Mathematical Knowing and Teaching," Lampert (1990) exemplifies her instructional design by describing a session devoted to discussion of the following problem: "What is the last digit in 5^4? 6^4? 7^4?" In order to illustrate the ways in which Sandy and I were inspired by her example and to indicate the sorts of discussions that occurred when we followed a similar format in our high school class, I will describe a class session in which we asked students to read from graphs (see Figure 4.1).

We asked students to tackle these questions early in their study of how the Cartesian coordinate system can be used to represent relationships between two quantities. Our students had used sketches of graphs to illustrate how the relationships between quantities changed. They had used more precise graphs to record information from tables and algebraic rules. But this was the first time they were being asked to use specific qualitative graphical information to make quantitative comparisons. They had no algorithm for producing a response; using their understanding of the nature of the Cartesian coordinate system, they would have to decide which aspects of the graph should be used to help answer each of the questions.

Although on the surface there are single, correct answers to each of the

Figure 4.1. An Algebra One problem.

Here are the graphs for the rates for four **different** pumps, which are emptying four **different** pools.



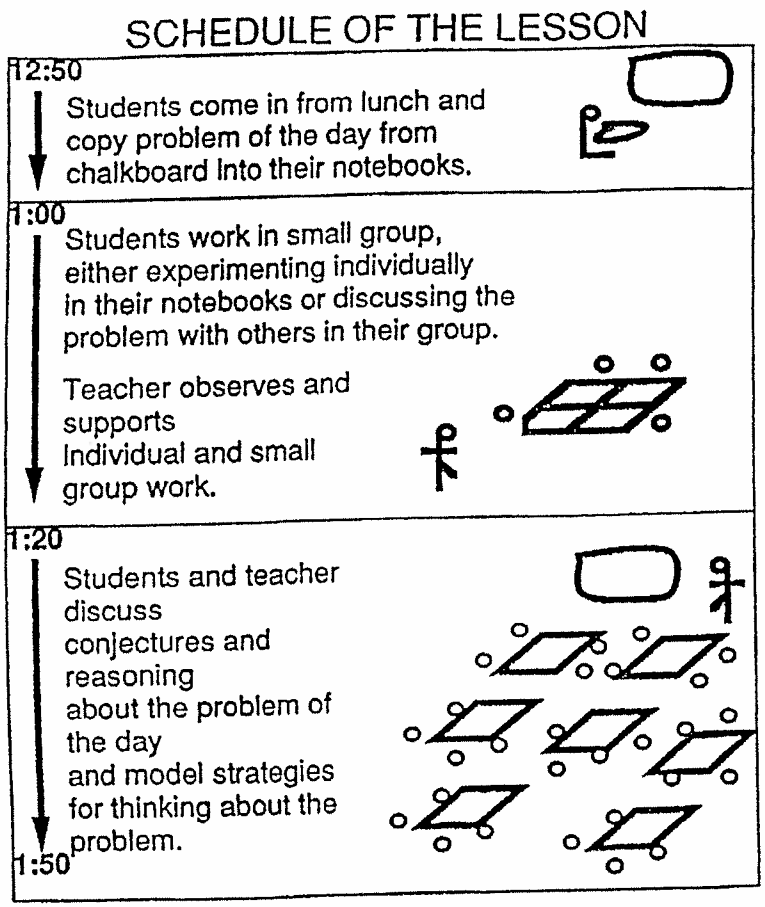
Explain all of your answers!
1. Which pool had the most water to start with?
2. Which pump completed its task first?
3. Which pump pumps the most water in a given time?

three questions, as in Lampert's (1990) instruction, our questions "put the solver in the position of devising all or part of the solution procedure" (p. 40). Our interest, like Lampert's, was not in the answers to the particular questions we asked, but, rather, in the nature of the rationales that students gave for their answers and the development of these rationales. Students' explanations for their choices would help us learn about their understanding of the use of the Cartesian coordinate system for representing relationships between two quantities.

Typically, after allowing students time for exploration, one of Lampert's classes might spend 30 to 40 minutes discussing such a problem and the ways in which people are thinking about the solution to the problem (see Figure 4.2).

While some might argue that such an emphasis on whole-group discussion forces students of different abilities to move at the same "pace," the beauty of this commitment of time to group discussion is that it can support the shift away from the teacher as sole arbiter of mathematical truth and toward the class as a mathematical community. In-depth discussion of problems can bring out the different ways in which students understand a problem, some of them mathemat-

Figure 4.2. The schedule of a typical lesson.



Source: Lampert (1991a)

ically correct and others mathematically incorrect. When a range of views is available to the class, students can then use logic and mathematical evidence to decide which answers and approaches they trust and value.

But orchestrating these whole-class discussions of a problem is not easy; good discussions do not occur spontaneously. In particular, such whole-group discussions require that students articulate their mathematical points of view clearly enough that the rest of the class can understand. The atmosphere in the class must help students not feel bad for offering "wrong" answers, but instead feel that such answers are steps along the way to the development of a deeper understanding—that offering wrong answers is part of the role of the student. Then, for the teacher, the challenge of leading the discussion is to keep students' ideas developing and growing toward deeper understanding.[18] When there are disagreements, the classroom community, the teacher included, has to support students in revising their initial conjectures.

The session in our class devoted to the discussion of the four-pump problem illustrates the nature of conversations when Sandy and I did not take on the role of arbiter of mathematical truth and instead left students to engage with each other's ideas. The students all chose pump A for the first question, seemingly for similar reasons. However, when we moved to the second question (which pump finished first?), it became clear that there were three reasons for choosing D, some correct and others incorrect. Jackie indicated that D was "first" on the y-axis; Christin chose D because it was the shortest line from y-intercept to x-intercept; and Bob thought D was the closest to (0,0), along the time axis.

As a result of seeing these different rationales for the same answer, students took it on themselves to decide which of these rationales would in general be correct. As part of that discussion, Bob argued that Christin's reasoning was not sound and attempted to develop an example that would distinguish his reasoning from hers (in my judgment, he was ultimately not successful). He hoped that doing so would provide mathematical evidence that would convince her that her rationale was not, in general, correct.

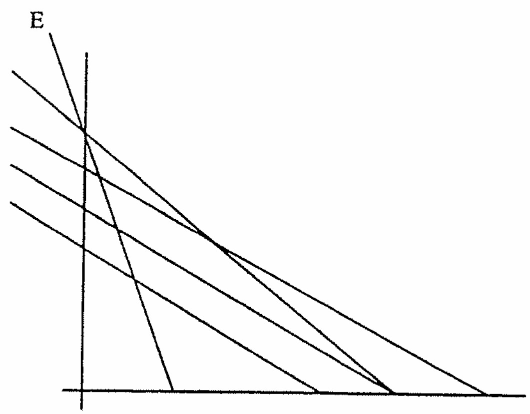
Bob: You can put something before this [D], it's a longer line, but still it's done first. So it doesn't really matter. Like watch. . . . Okay, like that [draws in line E; see Figure 4.3]. This [E] is longer than this one [D], but it [E] got done first.

Christin: That's not in it [the problem] though.

DC: So Bob has made up another graph, this one here, we can call it "E." (C: My point . . . [inaudible]) We can call it "E," and he says, Christin, that it's a counterexample to what you say. Joe?

Joe: It may not be an example in this time [this problem?], but still it's

Figure 4.3. Bob's pump E.



an example that you can't use that rule all the time that "it's the shortest one."

We moved to the third question, which I thought was about the "efficiency" of the pumps, but which students interpreted as deciding which pump was best in some sense. There were different answers, not only different reasons for the same answer. To some degree, these different answers reflected different views of the question.

Mark: I think number three's A because it has the most water, even though it had the most water, no matter what place it came in, it would always pump the most water.

Angela: I think it's D. . . . Because it's the one who got done first. I mean, it pumps, even though it has less water than everybody else's, . . . I think it was a better pump and it pumped faster.

Joe: A is the best, that has the most amount of water, and it still finished first, so no matter what you do it's just better, it's going to pump the most water at any given time.

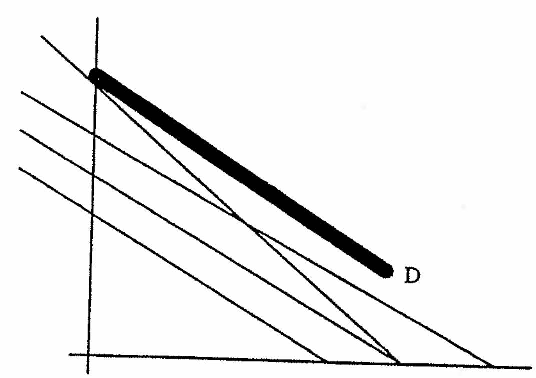
However, as students discussed these three answers, they did not address directly the different interpretations of the question that their answers reveal. Instead, adherents of the third response tried to indicate how their response was different from the other two. In what seemed to be the most convincing turn,

Bob again came to the board and presented an argument that asked the other students to compare the different pumps by doing a thought experiment and changing aspects of the given problem.

Bob: The way I look at it, you've got to look at the angles. Now say [Angela: Here's our mathematician!] say, take angle D [Bob places a ruler over line D] and put it up here [moves the left-hand edge of the ruler to the *y*-intercept of A, while keeping it parallel to D. See Figure 4.4]. This [A] still has a steeper angle, it will be done first. Take angle C, do the same thing, A has a steeper angle again, still, and do the same thing for this one [B], and it still has a steeper angle, so this is always going to get done first because it has a steeper angle than the rest.

This class session fascinates me. On the one hand, it illustrates connections between changes in the role of a problem (Lampert, 1990), the shifts in classroom roles proposed by NCTM (1991), and the nature of classroom conversation. Students do not have an algorithm for solving these questions. As a result, attention is focused less on the answers to the questions and more on the rationales for student responses. For example, even though on the second question everyone agrees that the answer is D, students still discuss the various rationales presented. In doing so, they make use of logic and evidence to attempt to convince each other to revise their points of view. In this discussion, students speak

Figure 4.4. Bob's use of the ruler.



to each other and take longer conversational turns than those common in most classrooms, and some students, like Bob, come to the board to represent their thinking in that public space. There is conversation in the mathematics classroom![17]

On the other hand, some of the tensions and complexities of our classroom are just below the surface. What does Angela's comment about Bob—"Here comes our mathematician"—mean? Is Angela expressing anger because Bob is arguing against her point of view? Is there an issue of gender that underlies the ways in which Bob argues against Angela and Christin's point of view? On the other hand, Angela, Bob, and Christin are all smokers. They tend to care about each other's point of view in class. Would Bob have bothered to argue against Christin's rationale if instead it had been articulated by a preppie? Is this why Angela calls him "our" mathematician? And what about the rationales students give? Are these articulated well enough to convince others?

BEYOND THE ROLE OF THE PROBLEM: TACKLING "RIGHT" AND "WRONG"

There is more to Lampert's (1990) teaching (and to the example I have just given) than reworking the pedagogical role of problems. Lampert's teaching is guided by mathematician George Polya (1945) and philosopher Imre Lakatos' (1976) descriptions of mathematics as a discipline, descriptions that differ from the conceptions of mathematics I have outlined earlier in this chapter. Simply changing the role of the problem is not enough. It is not enough for teachers to refrain from judging students' ideas. In line with Cuban's (1993) argument, I believe that one's beliefs about the nature of knowledge are central. In the case of mathematics, the categories of "right" and "wrong" have to be reexamined.

To illustrate, I'll return to the Algebra One classroom. In my first year of teaching at Holt, as I tried to create a student-centered classroom, a disquiet developed for me, focused on my role in evaluating students' comments and strategies. On the one hand, when I thought something was right or wrong, it seemed disingenuous to refrain from commenting, to withhold my views, particularly when students, concerned about knowing right from wrong, asked me directly. (These issues are explored in Chazan & Ball, 1999.) While I wanted to respect students' mathematical ideas, I did not want to retreat into an every-idea-is-of-equal-value brand of relativism; I did not want my students "taught" incorrect ideas.

On the other hand, I had three concerns about how to indicate, even subtly, that a comment or strategy was wrong. First, I was concerned that, if I told students flat out that they were wrong, they might give up and decide they were not able to make progress. Unlike students who have been successful in school and who have a strong belief in the value of school knowledge, my students were not goaded by a negative evaluation to return to their work and puzzle out where they had "gone wrong." Second, though I thought there was value in examining students' incorrect ideas, I was concerned that if I labeled an idea wrong, my students would no longer want to spend time on it. With my students' prior experiences of schooling, how would I justify spending time on something that was wrong? Finally, I was concerned I might undercut the dynamics leading to classroom conversation. If I simply told students what was right and wrong, why would they bother to listen to each other? After all, as the teacher, my judgment should be trustworthy, while students were in this class because they had not done well in mathematics. If I was going to tell them what was right and what was wrong, then wouldn't it be simpler and more efficient for me to do this without forcing them to work so hard? But, if I did that, I would be back in the role I had when teaching at the dayschool!

Propelled by these sorts of issues in the Holt teaching, I felt that I had to rethink my understandings of right and wrong and how I would present these categories to students. In this rethinking, I have drawn on recent debates about the nature of mathematical knowledge.[19]

CRITICISMS OF MATHEMATICAL CERTAINTY AND THEIR RAMIFICATIONS

Views of the nature of mathematical knowledge are becoming subject to the same sorts of examination as views of other kinds of knowledge (for example, the examinations of science associated with Thomas Kuhn, 1962, and of knowledge more generally associated with Richard Rorty, 1991). As a result, there are controversial challenges to the set of attitudes about mathematics that I outlined earlier. Most mathematicians would still characterize mathematics as "a field where things are either right or wrong . . . [where] the decisions are usually clean and straightforward" (Devlin, 1997, p. 2). But, for me, these challenges suggest that the edges around this set of conceptions are fraying. In my view, these challenges are resources for the development of new instruction in secondary school mathematics classrooms; with the reexamination of the nature of mathematical knowledge come opportunities to rework educational practices central to the teaching of mathematics.

At the heart of these critiques, the very notion of mathematical certainty, of the special character of mathematical knowledge, is being questioned and reevaluated. Critics of mathematical certainty, *pace* Devlin (1997), argue that labeling mathematical statements unambiguously right or wrong is not always a straightforward affair; the existence of a mathematical proof does not necessarily decide the matter once and for all. They argue that overdrawing the differ-